

# Dynamical Correspondence in a Generalized Quantum Theory

Gerd Niestegge

Fraunhofer ESK, Hansastr. 32, 80686 München, Germany  
gerd.niestegge@esk.fraunhofer.de, gerd.niestegge@web.de

*Abstract.* In order to figure out why quantum physics needs the complex Hilbert space, many attempts have been made to distinguish the  $C^*$ -algebras and von Neumann algebras in more general classes of abstractly defined Jordan algebras (JB- and JBW-algebras). One particularly important distinguishing property was identified by Alfsen and Shultz and is the existence of a dynamical correspondence. It reproduces the dual role of the selfadjoint operators as observables and generators of dynamical groups in quantum mechanics. In the paper, this concept is extended to another class of nonassociative algebras, arising from recent studies of the quantum logics with a conditional probability calculus and particularly of those that rule out third-order interference. The conditional probability calculus is a mathematical model of the Lüders-von Neumann quantum measurement process, and third-order interference is a property of the conditional probabilities which was discovered by R. Sorkin in 1994 and which is ruled out by quantum mechanics. It is shown then that the postulates that a dynamical correspondence exists and that the square of any algebra element is positive still characterize, in the class considered, those algebras that emerge from the selfadjoint parts of  $C^*$ -algebras equipped with the Jordan product. Within this class, the two postulates thus result in ordinary quantum mechanics using the complex Hilbert space or, vice versa, a genuine generalization of quantum theory must omit at least one of them.

*Key Words.* Order derivations, positive groups, operator algebras, Lie algebras, foundations of quantum mechanics

*PACS.* 03.65Fd, 03.65Ta, 02.30Tb

*MSC.* 46L70, 81P10

## 1 Introduction

In quantum mechanics, the selfadjoint operators play a dual role; they represent observables - the measurable physical quantities of the system under consideration - as well as generators of dynamical groups - describing the time evolution of the system. In a more general setting, this dual role is reproduced by a dynamical correspondence allocating a generator of a dynamical group on an ordered Banach space to each element of the space; in this case, the group generators are skew order derivations. Dynamical correspondences were introduced by Alfsen and Shultz [1, 3] as a mean to distinguish the selfadjoint parts of the  $C^*$ -algebras

and von Neumann algebras among Jordan algebras of a more general type - the JB- and JBW-algebras - and to figure out why quantum physics needs the complex Hilbert space. Dynamical correspondences base upon Connes' notion of order derivations [7].

In the present paper, which is a sequel of Ref. [13], the notion of a dynamical correspondence is extended to another class of nonassociative algebras comprising the JBW algebras and arising from recent studies of the quantum logics with a conditional probability calculus and particularly of those that rule out third-order interference [11, 12, 13]. The conditional probability calculus is a mathematical model of the Lüders-von Neumann quantum measurement process. Third-order interference is a property of the conditional probabilities which was discovered by R. Sorkin in 1994 and which is ruled out by quantum mechanics [14].

Below, it will be shown that the postulates that a dynamical correspondence exists and that the square of any algebra element is positive still characterize, in the class considered, those algebras that emerge from the selfadjoint parts of  $C^*$ -algebras equipped with the Jordan product.

The class of nonassociative algebras is defined in section 2 and, how operator algebras and Jordan algebras fit into this setting, is explained in section 3. Order derivations are considered in section 4, before turning to the dynamical correspondences and the main results in section 5.

## 2 The ordered Banach algebra

Let  $A$  be a complete order-unit space with distinguished order-unit  $\mathbb{I}$  [3]. Its unit interval  $[0, \mathbb{I}] := \{a \in A : 0 \leq a \leq \mathbb{I}\} = \{a \in A : 0 \leq a \text{ and } \|a\| \leq 1\}$  is a convex set. As usual, an element in a convex set is called an extreme point of this set if it is not any convex combination of two other elements in this set. The set of extreme points of the unit interval is denoted by  $\text{ext}[0, \mathbb{I}]$ . It includes the order-unit  $\mathbb{I}$  and forms a quantum logical structure with the orthogonality relation  $e \perp f : \Longleftrightarrow e + f \in \text{ext}[0, \mathbb{I}]$  and with the orthocomplementation  $e' := \mathbb{I} - e$  ( $e, f \in \text{ext}[0, \mathbb{I}]$ ). Its elements (the extreme points of  $[0, \mathbb{I}]$ ) are called *events*.

A *state*  $\mu$  on this quantum logic allocates the probability  $\mu(f) \in [0, 1]$  to each event  $f$  and is an orthogonally additive function from  $\text{ext}[0, \mathbb{I}]$  to the non-negative real numbers with  $\mu(\mathbb{I}) = 1$ . The *conditional probability* of an event  $f$  under another event  $e$  in the state  $\mu$  with  $\mu(e) \neq 0$  is the updated probability for  $f$  after the outcome of a first measurement has been the event  $e$ ; it is denoted by  $\mu(f|e)$ . Mathematically, it is defined by the conditions that the map  $\text{ext}[0, \mathbb{I}] \ni f \rightarrow \mu(f|e)$  is a state on  $\text{ext}[0, \mathbb{I}]$  and that the identity  $\mu(f|e) = \mu(f)/\mu(e)$  holds for all events  $f \in \text{ext}[0, \mathbb{I}]$  with  $f \leq e$ .

Note below that an operator  $S : A \rightarrow A$  (or a function  $\rho : A \rightarrow \mathbb{R}$ ) is called *positive* if  $S(a) \geq 0$  ( $\rho(a) \geq 0$ ) for all  $a \in A$  with  $a \geq 0$ .

In the remaining part of this paper, it shall be assumed that  $A$  is a complete order-unit space with order-unit  $\mathbb{I}$  and that the following four conditions (A), (B), (C) and (D) are satisfied:

- (A) There is a bilinear multiplication  $\square$  on  $A$  with  $\mathbb{I}$  as multiplicative identity,  $\|a\square b\| \leq \|a\| \|b\|$  for  $a, b \in A$  and  $e^2 := e\square e = e$  for all  $e \in \text{ext}[0, \mathbb{I}]$ .
- (B) Define  $T_a b := a\square b$  for  $a, b \in A$  and  $U_e := 2T_e^2 - T_e$  for  $e \in \text{ext}[0, \mathbb{I}]$ . Each linear operator  $U_e$  is positive and its range  $U_e A$  is the closed linear hull of  $\{f \in \text{ext}[0, \mathbb{I}] \mid f \leq e\}$ .
- (C) Every state  $\mu$  on the quantum logic  $\text{ext}[0, \mathbb{I}]$  has a positive linear extension on  $A$  which is again denoted by  $\mu$  (Note that this extension is unique since the positive linear functionals on an order-unit space are continuous and since, by applying (B) to  $e = \mathbb{I}$ ,  $A$  is the closed linear hull of  $\text{ext}[0, \mathbb{I}]$ . This also implies that  $[0, \mathbb{I}]$  contains sufficiently many extreme points).
- (D) If  $\mu$  is a state and  $0 \leq a \in A$  with  $\mu(a) = 0$ , then  $\mu(a\square b) = 0$  for all  $b \in A$ .

Note that, generally, the product  $\square$  is neither commutative nor associative,  $\mu(b\square a) = 0$  does not hold in condition (D), and the square  $a^2 = a\square a$  of an element  $a \in A$  is not positive.

Lemma 1: Suppose  $e, f \in \text{ext}[0, \mathbb{I}]$  and  $a \in A$ ; then:

- (i)  $U_e^2 = U_e$  and  $U_e U_{e'} = U_{e'} U_e = 0$ .
- (ii)  $U_e f = f$  and  $U_{e'} f = 0$  for  $f \leq e$ . Moreover,  $U_e f = 0$  and  $U_{e'} f = f$  for  $e \perp f$ .
- (iii)  $e\square a = T_e a = (a + U_e a - U_{e'} a)/2$ .
- (iv)  $e\square f = T_e f = f$  and  $e'\square f = T_{e'} f = 0$  for  $f \leq e$ . Moreover,  $e\square f = T_e f = 0$  and  $e'\square f = T_{e'} f = f$  for  $e \perp f$ .

Proof. (i) Assume that  $\mu$  is a state with  $\mu(e) \neq 0$ . Then  $\nu(f) := \mu(U_e f)/\mu(e)$  for the events  $f$  defines a state  $\nu$ . The identity  $e^2 = e$  implies  $\nu(e) = 1$  and thus  $\nu(e') = 0$ . By (D),  $0 = \nu(e'\square x) = \mu(U_e(e'\square x))/\mu(e)$  and thus  $0 = \mu(U_e(e'\square x))$  for any  $x$  in  $A$ .

If  $\mu(e) = 0$ , then  $0 = \mu(T_e y)$  for all  $y \in A$  by (D) and  $0 = \mu(U_e y)$  by the definition of  $U_e$ . Thus again  $0 = \mu(U_e(e'\square x))$  for any  $x \in A$ .

Therefore  $0 = U_e(e'\square x)$  for any  $x \in A$ . This means  $0 = U_e T_{e'} = U_e - U_e T_e$  and  $U_e = U_e T_e$ . Using again the definition of  $U_e$  and  $U_{e'}$  finally gives  $U_e^2 = U_e(2T_e^2 - T_e) = U_e$  and  $U_e U_{e'} = U_e(2T_{e'}^2 - T_{e'}) = 0$ . Replacing  $e$  by  $e'$  yields  $U_{e'} U_e = 0$ .

- (ii) Suppose  $f \leq e$ . Then  $U_f A \subseteq U_e A$  by (B). Therefore  $U_e U_f = U_f$  by (i) and  $U_e f = U_e U_f \mathbb{I} = U_f \mathbb{I} = f$ . Moreover,  $0 \leq U_{e'} f \leq U_{e'} e = 0$  and hence  $U_{e'} f = 0$ . If  $e \perp f$ , then  $f \leq \mathbb{I} - e = e'$  and, replacing  $e$  by  $e'$  in the first part of (ii), yields  $U_e f = 0$  as well as  $U_{e'} f = f$ .

(iii) Note that  $U_{e'} a = 2e'\square(e'\square a) - e'\square a = 2(\mathbb{I} - e)\square(e'\square a) - e'\square a = 2(e'\square a) - 2e\square(e'\square a) - e'\square a = e'\square a - 2e\square(e'\square a) = a - e\square a - 2e\square a + 2e\square(e\square a) = a - 3T_e a + 2T_e^2 a$  and therefore,  $a + U_e a - U_{e'} a = a + 2T_e^2 a - T_e a - a + 3T_e a - 2T_e^2 a = 2T_e a$ .

(iv) follows from (ii) and (iii). ■

Lemma 2: With the above assumptions, the conditional probability  $\text{ext}[0, \mathbb{I}] \ni f \rightarrow \mu(f|e)$  exists and is uniquely determined for any state  $\mu$  and event  $e$  with  $\mu(e) \neq 0$ . Moreover,  $\mu(f|e) = \mu(U_e f)/\mu(e)$ .

Proof. Suppose  $\mu(e) \neq 0$  and define  $\nu_1(f) := \mu(U_e f)/\mu(e)$  for  $f \in \text{ext}[0, \mathbb{I}]$ . By Lemma 1 (ii),  $\nu_1$  is a conditional probability.

Assume that  $\nu_2$  is a second conditional probability under  $e$  in the state  $\mu$ . Then  $\nu_2(e) = 1$  and  $\nu_2(e') = 0$ . By (D),  $\nu_2(e' \square x) = 0$  for all  $x$  in  $A$  or, equivalently,  $\nu_2(T_e x) = \nu_2(e \square x) = \nu_2(x)$ . This means that  $\nu_2$  is invariant under  $T_e$  and therefore under  $U_e = 2T_e^2 - T_e$ .

For any  $x \in A$  then  $\nu_2(x) = \nu_2(U_e x)$ . Since  $U_e x$  lies in the closed linear hull of  $\{f \in \text{ext}[0, \mathbb{I}] \mid f \leq e\}$ , the characteristics of the conditional probability and the continuity and linearity of  $\nu_2$  imply  $\nu_2(x) = \nu_2(U_e x) = \mu(U_e x)/\mu(e) = \nu_1(x)$ . Therefore,  $\nu_1 = \nu_2$ . ■

The mathematical structure defined in this section has two important aspects. On the one hand, the next section will show that it covers the operator algebras used in quantum physics, but is more general.

On the other hand, it stems from recent studies of the quantum logics with a conditional probability calculus (i.e. with a reasonable model of the Lüders-von Neumann quantum measurement process) and particularly of those that rule out third-order interference [12, 13]. It can thus be regarded as a generalized quantum theory. However, note that it does not encompass the most general case studied in Refs. [12, 13]. For instance, generally, the quantum logic need not coincide with the extreme points of the unit interval, and condition (D) is not satisfied for all positive elements  $a$  in  $A$ , but only if  $a$  lies in the quantum logic. Moreover, in the infinite-dimensional case, the norm topology and certain weak topologies must be distinguished and the norm topology must be replaced by a weak topology in some cases to cover the most general situation.

### 3 Operator algebras

The formally real Jordan algebras were introduced in 1934 by Jordan, von Neumann and Wigner [9]. Forty years later, this theory was extended by Alfsen, Shultz, Størmer and others to include infinite dimensional algebras; these are the so-called JB-algebras and JBW-algebras. The monograph [3] contains a comprehensive representation of the theory of the JB-/JBW-algebras. The selfadjoint part of a C\*-algebra, equipped with the Jordan product  $a \circ b = (ab + ba)/2$ , is a JB-algebra, and the selfadjoint part of a von Neumann algebra is a JBW-algebra. All these algebras are order-unit spaces.

A JBW-algebra  $A$  without type  $I_2$  part satisfies the four conditions in section 2, with the Jordan product  $\circ$  playing the role of the  $\square$ -product. Condition (A) is the well-known fact that the extreme points of the positive part of the unit ball in a JBW-algebra coincide with the idempotent elements of the JBW-algebra [3]. The positivity of the  $U_e$  is a rather non-trivial result in the theory of the JBW-algebras [3] and, together with the spectral theorem, it implies the remaining part of condition (B). Condition (C) is the extension of Gleason's theorem to JBW-algebras which holds iff  $A$  does not contain a type  $I_2$  part [6]. Condition (D) follows from the Cauchy-Schwarz inequality for the positive

linear functionals and the fact that, by the spectral theorem,  $\mu(a) = 0$  implies  $\mu(a^2) = 0$  for  $0 \leq a$ . A more explicit elaboration of these considerations can be found in [10].

In the specific situation where  $A$  is the selfadjoint part of a von Neumann algebra  $M$ , the operators  $U_e$  get the familiar shape  $U_e a = eae$  ( $a, e \in A$  and  $e^2 = e$ ), which also reveals the link to the Lüders-von Neumann quantum measurement process. In this case, the positivity of  $U_e$  follows from  $U_e a = eae = (a^{1/2}e)^*(a^{1/2}e) \geq 0$  for  $a \geq 0$ .

## 4 Order derivations

A bounded linear operator  $D : A \rightarrow A$  is called an *order derivation* if  $e^{tD}$  is positive for any real number  $t$ . The order derivations are generators of positive groups. They were introduced by Connes [7]. Most interesting are those positive groups, which leave the order-unit invariant for all  $t$ , since they entail automorphism groups of the state space [13]; this holds when the generator  $D$  satisfies the condition  $D(\mathbb{I}) = 0$ . Such a derivation  $D$  describes the dynamical evolution satisfying the simple linear differential equation  $\frac{d}{dt}x_t = Dx_t$  ( $x_t \in A$ ). Any physical theory with a reversible time evolution should include one-parameter automorphism groups and therefore at least some derivations  $D$  with  $D(\mathbb{I}) = 0$ . Generally, they need not be bounded, but note that only bounded derivations are considered in this paper.

The following lemma provides a very useful general characterization of order derivations; it was first used by Connes in a more specific context [7] and then generalized by Evans and Hanche-Olsen [8] (see also [2]).

**Lemma 3.** Let  $D$  be a bounded linear operator from  $A$  into  $A$ . Then the following are equivalent:

- (i)  $D$  is an order derivation.
- (ii) If  $\mu$  is a state and  $0 \leq x \in A$  with  $\mu(x) = 0$ , then  $\mu(Dx) = 0$ .

By Lemma 3 and condition (D) in section 2, the right-hand side multiplication operators  $R_a$  with  $R_a x := x \square a$ ,  $x \in A$ , are order derivations for all  $a \in A$ . They are called *selfadjoint*. Of course,  $R_a(\mathbb{I}) = a$ . The more interesting order derivations  $D$  with  $D(\mathbb{I}) = 0$  are called *skew*. Any order derivation  $D$  is the sum of a selfadjoint order derivation  $D_1$  and a skew order derivation  $D_2$ ; with  $a := D(\mathbb{I})$  choose  $D_1 := R_a$  and  $D_2 := D - D_1$ .

This naming (selfadjoint and skew) stems from the fact that, in a von Neumann algebra, the selfadjoint order derivations have the shape  $D(x) = (ax + xa)/2$  and the skew order derivations have the shape  $D(x) = i(bx - xb)/2$ , where  $a, b$  are selfadjoint elements in the von Neumann algebra [3].

The commutator  $[D_1, D_2] := D_1 D_2 - D_2 D_1$  of any two order derivations  $D_1$  and  $D_2$  is an order derivation again and the order derivations thus form a Lie algebra [2]. It is obvious that the commutator is skew if  $D_1$  and  $D_2$  are skew. Therefore the skew order derivations form a Lie subalgebra  $L$ ; its elements are generators of one-parameter automorphism groups which describe reversible

dynamical evolutions. With any pair of elements  $a$  and  $b$  in the order-unit space  $A$ , the operator  $[R_a, R_b] - R_{b \square a - a \square b}$  lies in the Lie algebra  $L$ .

The next two lemmas provide some useful algebraic properties of the order automorphisms and skew order derivations (similar to the situation in the more specific JB-algebras [3]).

Lemma 4: (i) If  $W : A \rightarrow A$  is an order automorphism with  $W(\mathbb{I}) = \mathbb{I}$ , then  $W(a \square b) = W(a) \square W(b)$  for any  $a, b \in A$ .

(ii) If  $D$  is a skew order derivation on  $A$ , then  $D(a \square b) = D(a) \square b + a \square D(b)$  for any  $a, b \in A$ .

*Proof.* (i) Suppose that  $W$  is an order automorphism on  $A$  with  $W(\mathbb{I}) = \mathbb{I}$ . Then  $W$  maps  $\text{ext}[0, \mathbb{I}]$  onto itself.

For any state  $\mu$  on  $\text{ext}[0, \mathbb{I}]$ ,  $\mu W : e \rightarrow \mu(W(e))$  is a state on  $\text{ext}[0, \mathbb{I}]$  and  $\mu(W(\cdot)|W(e))$  is a conditional probability under the event  $e$  in the state  $\mu W$ . Its uniqueness (Lemma 2) implies that  $\mu(W(f)|W(e)) = \mu W(f|e)$  for any  $e, f \in \text{ext}[0, \mathbb{I}]$  with  $\mu(W(e)) > 0$ . That is  $\mu(U_{W(e)}W(f)) = \mu(W(U_e f))$ . If  $\mu(W(e)) = 0$ , then  $0 \leq \mu(U_{W(e)}W(f)) \leq \mu(U_{W(e)}\mathbb{I}) = \mu(W(e)) = 0$  and  $0 \leq \mu(W(U_e f)) \leq \mu(W(U_e \mathbb{I})) = \mu(W(e)) = 0$ . Thus  $\mu(U_{W(e)}W(f)) = 0 = \mu(W(U_e f))$ .

Therefore,  $W(U_e f) = U_{W(e)}W(f)$  for any events  $e$  and  $f$ ; by Lemma 1 (iii), it follows that  $W(e \square f) = W(e) \square W(f)$ . Since  $A$  is the closed linear hull of the events, the continuity and linearity of the product and of  $W$  finally imply  $W(a \square b) = W(a) \square W(b)$  for  $a, b \in A$ .

(ii) Suppose that  $a, b \in A$  and that  $D$  is a skew order derivation. By part (i)  $e^{tD}(a \square b) = e^{tD}(a) \square e^{tD}(b)$  for all real numbers  $t$ . Differentiating both sides of this equation at  $t = 0$  gives  $D(a \square b) = D(a) \square b + a \square D(b)$ . ■

Lemma 5:  $[D, R_a] = R_{D(a)}$  for any skew order derivation  $D$  and  $a \in A$ .

*Proof.* Suppose  $a, x \in A$ . Then  $D(x \square a) = D(x) \square a + x \square D(a)$  by Lemma 4. This can be rewritten as  $DR_a x = R_a D x + R_{D(a)} x$ . ■

## 5 Dynamical correspondence

In a von Neumann algebra  $M$ , there is the following one-to-one correspondence  $a \rightarrow D_a$  between the selfadjoint elements  $a$  of the algebra and the skew order derivations [3]:  $D_a x = i(ax - xa)/2$  for  $x \in M$ . In this case,  $[D_a, D_b] = -[R_a, R_b]$  and  $D_a a = 0$  for all selfadjoint  $a, b \in M$ . Moreover, this specific correspondence distinguishes those JB- and JBW-algebras that are the selfadjoint parts of C\*- and von Neumann algebras from the other ones. This motivates the following definition which is due to Alfsen and Shultz [1, 3], but adapted to the more general setting of this paper. Alfsen and Shultz consider only the JB- and JBW-algebras; since these are commutative, they need not distinguish between the right-hand side and left-hand side multiplication operators  $R_a$  and  $T_a$ .

**Definition 1.** A *dynamical correspondence* is a linear map  $a \rightarrow D_a$  from  $A$  into the Lie algebra  $L$  of skew order derivations on  $A$ , which satisfies the following two conditions:

- (i)  $[D_a, D_b] = -[R_a, R_b]$  for  $a, b \in A$ ,
- (ii)  $D_a a = 0$  for all  $a \in A$ .

Condition (i) links the dynamical correspondence to the multiplication operation  $\square$  and immediately implies its commutativity. Applying to  $\mathbb{I}$  both sides of the equation gives  $0 = b\square a - a\square b$ . Therefore, the operators  $R_a$  and  $T_a$  become identical. Condition (i) thus has important mathematical consequences, but lacks any physical justification. Condition (ii) means that  $a$  is invariant under the one-parameter dynamical group generated by  $D_a$ .

It shall now be seen that the existence of a dynamical correspondence implies not only the commutativity, but also the power-associativity and the Jordan property of the product. The following lemma and theorem and the proofs are transfers of the results in [3] for JBW-algebras to the more general setting of this paper. Lemma 4 (ii) is the key making this possible.

Lemma 6: Assume that the map  $a \rightarrow D_a$  from  $A$  into  $L$  is a dynamical correspondence. Then

- (i)  $[D_a, R_b] = [R_a, D_b]$  and
- (ii)  $D_a b = -D_b a$  for any  $a, b \in A$ .

Proof. (i) Assume that the map  $a \rightarrow D_a$  is a dynamical correspondence. By Lemma 5 and condition (ii) of Definition 1,  $[D_a, R_a] = R_{D_a a} = R_0 = 0$  for all  $a \in A$ . Therefore, by the linearity of the dynamical correspondence  $a \rightarrow D_a$ , for all  $a, b \in A$ ,  $0 = [D_{a+b}, R_{a+b}] = [D_a, R_b] + [D_b, R_a]$ . This gives  $[D_a, R_b] = -[D_b, R_a] = [R_a, D_b]$ .

(ii) Lemma 5 and (i) of Lemma 6 imply for all  $a, b \in A$  that  $D_a b = R_{D_a b} \mathbb{I} = [D_a, R_b] \mathbb{I} = [R_a, D_b] \mathbb{I} = -[D_b, R_a] \mathbb{I} = -R_{D_b a} \mathbb{I} = -D_b a$ . ■

**Theorem 1:** If  $A$  admits a dynamical correspondence, it is (isomorphic to) the selfadjoint part of an associative  $*$ -algebra over the complex numbers and the product  $\square$  becomes identical with the Jordan product:  $a\square b = a \circ b = (ab + ba)/2$  for  $a, b \in A$  (This means that  $A$  is a special Jordan algebra).

Proof. Assume that  $a \rightarrow D_a$  is a dynamical correspondence on  $A$ . By Lemma 6, an anti-symmetric bilinear product  $\times$  can be defined on  $A$  via  $a \times b := D_a b$  for  $a, b \in A$ . A further bilinear map into  $A + iA$  (considered as a real-linear space) can be defined on  $A$  via:  $ab := a\square b - i(a \times b)$ . This map can be uniquely extended to a bilinear product on  $A + iA$  (considered as a complex-linear space). It shall now be shown that this product is associative. Because of its linearity, it suffices to prove that  $a(cb) = (ac)b$  for  $a, b, c \in A$ . This means

$$\begin{aligned} & a\square(c\square b) - i(a \times (c\square b)) - i(a\square(c \times b)) - (a \times (c \times b)) \\ &= (a\square b)\square c - i((a\square b) \times c) - i((a \times c)\square b) - (a \times c) \times b. \end{aligned}$$

Separating real and imaginary terms and using the anti-symmetry of the  $\times$ -product yields the following two equations:

$$a \times (b \times c) - b \times (a \times c) = -a \square (b \square c) + b \square (a \square c)$$

and

$$a \times (b \square c) - b \square (a \times c) = a \square (b \times c) - b \times (a \square c).$$

The left-hand side of the first one of these two equations is just  $[D_a, D_b]c$  and its right-hand side is  $-[R_a, R_b]c$ . Note that the product  $\square$  is commutative and  $T_x = R_x$  for  $x \in A$ , which follows from the existence of the dynamical correspondence. Similarly the left-hand side of the second equation is  $[D_a, R_b]c$  and its right hand side is  $[R_a, D_b]c$ . Thus the first one of these two equations follows directly from Definition 1 and the second one from Lemma 6.

The involution on  $A + iA$  is defined by  $(a + ib)^* = a - ib$  and it must still be shown that  $(xy)^* = y^*x^*$  for  $x, y \in A$ . By linearity it suffices to prove that  $(ab)^* = ba$  for  $a, b \in A$ . This follows from the anti-symmetry of the  $\times$ -product, since

$$(ab)^* = (a \square b - i(a \times b))^* = a \square b + i(a \times b) = b \square a - i(b \times a) = ba.$$

Therefore,  $A + iA$  is an associative  $*$ -algebra and its selfadjoint part is  $A$ . Moreover,  $(ab + ba)/2 = a \square b$  for  $a, b \in A$ . Note that the last equation again requires the commutativity of the product  $\square$ . ■

**Corollary 1:** If  $A$  admits a dynamical correspondence and if  $a^2 \geq 0$  for any  $a$  in  $A$ , then  $A$  is (isomorphic to) the selfadjoint part of a  $C^*$ -algebra.

Proof. This follows immediately from Theorem 1 and Theorem 1.96 in Ref. [2] (or A59 in Ref. [3]); the  $C^*$ -norm on  $A + iA$  is given by  $\|x\| := \|x^*x\|^{1/2}$  for  $x$  in  $A + iA$ . ■

Theorem 1 shows that the assumption that a dynamical correspondence exists is very strong. From a general starting point, it immediately results in special Jordan algebras which, moreover, are the selfadjoint parts of  $*$ -algebras over the complex numbers; real algebras that cannot be obtained as selfadjoint parts of complex  $*$ -algebras are ruled out.

However, Theorem 1 does not yet lead to ordinary Hilbert space quantum mechanics. This is achieved by the additional assumption that the squares of the elements in  $A$  are positive. By Corollary 1,  $A$  is the selfadjoint part of a  $C^*$ -algebra then and, by the Gelfand-Naimark Theorem [2],  $A$  can be represented as operators on a complex Hilbert space. In doing so,  $A$  exhausts the full algebra of all operators on the Hilbert space in some cases or forms a genuine subalgebra (as physically required with the presence of superselection rules) in other cases.

Examples of algebras with positive squares, but without dynamical correspondences are the formally real Jordan algebras  $H_n(\mathbb{R})$  and  $H_n(\mathbb{H})$  ( $n \geq 3$ ) and the exceptional Jordan algebra  $H_3(\mathbb{O})$ . They consist of the hermitian  $n \times n$ -matrices over the real numbers ( $\mathbb{R}$ ), quaternions ( $\mathbb{H}$ ) or octonions ( $\mathbb{O}$ ) [3]. Ex-



amples with dynamical correspondences, but with non-positive squares are not known.

## 6 Conclusions

In order to figure out why quantum physics needs the complex Hilbert space, many attempts have been made to distinguish the  $C^*$ -algebras and von Neumann algebras from the more general JB- and JBW-algebras. Different distinguishing properties have been identified: dynamical correspondences, the 3-ball property and orientations [2, 3]. Only the dynamical correspondence has a certain physical meaning, since it establishes a relation between the algebra elements and the bounded generators of one-parameter dynamical groups. However, only the existence of (possibly unbounded) group generators can be considered an important requirement for any reasonable physical theory, and theories without dynamical correspondences or with less strong versions might be thinkable. Alfsen and Shultz's condition (i) in Definition 1 (section 5) links the dynamical correspondence to the multiplication operation  $\square$ . It is a mathematically nice and strong assumption, but lacks a proper physical justification. The *energy observable assignment* defined in [5] represents a weaker form of a dynamical correspondence dispensing with condition (i). It may be better justified from the physical point of view, but the mathematical methods applied in [5] fail in the infinite-dimensional case and it is not known whether the results in [5] remain valid in this case.

In the present paper, Alfsen and Shultz's definition of a dynamical correspondence has been used and it has been seen that this notion can be extended to a class of nonassociative algebras, which is much broader than the JBW algebras. This class arises from recent studies of the quantum logics with a conditional probability calculus (i.e., with a reasonable model of the Lüders - von Neumann quantum measurement process) and particularly of those that rule out third-order interference. The existence of a dynamical correspondence for an algebra in this class still entails that it is the selfadjoint part of a  $C^*$ -algebra, if it is assumed that the squares of the algebra elements are positive (Corollary 1). The Jordan property of the product or its power-associativity become redundant requirements in this situation. The same holds for some other conditions used for abstract mathematical characterizations of operator algebras or their state spaces (e.g., spectrality and ellipticity [3]).

Thus, within the considered class of nonassociative algebras, the two postulates that a dynamical correspondence exists and that the square of any algebra element is positive result in ordinary quantum mechanics using the complex Hilbert space or, vice versa, a genuine generalization of quantum theory must omit at least one of them.

In section 4, it has been seen that the skew order derivations form a Lie algebra. Almost all finite-dimensional simple Lie algebras arise from the derivations on the finite-dimensional formally real Jordan algebras (the finite-dimensional version of the JB-/JBW-algebras), and there are only four exceptions ( $\mathfrak{g}_2$ ,  $\mathfrak{e}_6$ ,

$\mathfrak{e}_7$  and  $\mathfrak{e}_8$  [4]). An interesting question now becomes whether these four emerge from the skew order derivations on some unknown nonassociative algebras out of the class which is defined in the second section and comprises the finite-dimensional formally real Jordan algebras. If such a nonassociative algebra exists, it either contains elements with non-positive squares or does not possess a dynamical correspondence, and its continuous symmetries form one of the exceptional Lie groups  $G_2$ ,  $E_6$ ,  $E_7$  and  $E_8$ .

## References

- [1] Alfsen, E.M., and Shultz, F. W.: On orientation and dynamics in operator algebras (Part I). *Commun. Math. Phys.* **194**, 87-108 (1998)
- [2] Alfsen, E.M., and Shultz, F. W.: State spaces of operator algebras: basic theory, orientations and C\*-products. *Mathematics: Theory & Applications*, Birkhäuser, Boston (2001)
- [3] Alfsen, E.M., and Shultz, F. W.: Geometry of state spaces of operator algebras. *Mathematics: Theory & Applications*, Birkhäuser, Boston (2003)
- [4] Baez, J. C.: The octonions. *Bull. Amer. Math. Soc.* **39**, 145-205 (2001)
- [5] Barnum, H., Müller, M. P., and Ududec, C.: Higher-order interference and single-system postulates characterizing quantum theory. *New J. Phys.* **16**, 123029 (2014)
- [6] Bunce, L. J., and Wright J. D. M.: Continuity and linear extensions of quantum measures on Jordan operator algebras. *Math. Scand.* **64**, 300-306 (1989)
- [7] Connes, A.: Characterisation des espaces vectoriels ordonnés sous-jacent aux algèbres de von Neumann. *Ann. Inst. Fourier (Grenoble)* **24**, 121-155 (1974)
- [8] Evans, D., and Hanche-Olsen, H.: The generators of positive semigroups. *J. Func. Analysis* **32**, 207-212 (1979)
- [9] Jordan, P., von Neumann, J., and Wigner, E.: On an algebraic generalization of the quantum mechanical formalism. *Ann. Math.* **35**, 29-64 (1934)
- [10] Niestegge, G.: Non-Boolean probabilities and quantum measurement. *J. Phys. A* **34**, 6031-6042 (2001)
- [11] Niestegge, G.: A representation of quantum mechanics in order-unit spaces. *Found. Phys.* **38**, 783-795 (2008)
- [12] Niestegge, G.: Conditional probability, three-slit experiments, and the Jordan algebra structure of quantum mechanics. *Adv. Math. Phys.* **2012**, 156573 (2012)

- [13] Niestegge, G.: A generalized quantum theory. *Found. Phys.* **44**, 1216-1229 (2014)
- [14] Sorkin, R. D.: Quantum mechanics as quantum measure theory. *Mod. Phys. Lett. A* **9**, 3119-3127 (1994)